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Variational inequalities on weakly compact sets

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Abstract In this paper, we derive an existence result for generalized variational inequalities associated with multivalued mappings on weakly compact sets under a continuity assumption which is much weaker than the regular complete continuity. As an application, we prove the existence of exceptional families of elements for such mappings on closed convex cones in reflexive Banach spaces when the corresponding complementarity problems have no solutions.

Keywords Regular complete continuity · Generalized variational inequalities · Exceptional families of elements

1 Introduction

Throughout this paper, X denotes a normed space over $\mathbb{R}$ equipped with the norm $\|\cdot\|$. The set of all continuous linear functionals from X into $\mathbb{R}$ is written by X^*. For any given $x \in X$ and $y \in X^*$, we shall write the value of y at x as $\langle y, x\rangle$.

Let 2^{X^*} denote the set of all subsets of X. A mapping T from a nonempty set $\Omega \subset X$ into 2^{X^*} will be called a multivalued mapping from Ω into X^*. The graph of T is defined by

$$\mathcal{G}_T = \{(x, y) : x \in \Omega \text{ and } y \in T(x)\}.$$

The generalized variational inequality GVI(T, Ω) associated with a multivalued mapping T from a closed convex set $\Omega \subset X$ into X^* is the problem to find a pair $(\widehat{x}, \widehat{y}) \in \mathcal{G}_T$ such that

$$\langle \widehat{y}, u - \widehat{x}\rangle \geq 0 \quad \text{for all} \quad u \in \Omega.$$

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Such a pair $(\widehat{x}, \widehat{y})$ will be called a solution of the problem GVI(T, Ω). We shall say that the problem GVI(T, Ω) is solvable if the problem has a solution. When $T(x)$ consists of a single element of X^* for every $x \in X$, the mapping T is called single valued, and in this case, the problem GVI(T, Ω) becomes the variational inequality problem VI(T, Ω).

If Ω is a closed convex cone in X, then the corresponding generalized variational inequality problem is usually called the multivalued complementarity problem, written by MCP(T, Ω). It is well-known that a pair $(\widehat{x}, \widehat{y}) \in \mathcal{G}_T$ is a solution of the problem MCP(T, Ω) if and only if $\widehat{y} \in \Omega^*$ and $\langle \widehat{y}, \widehat{x} \rangle = 0$, where

$$\Omega^* = \{y \in X^* : \langle y, x \rangle \geq 0 \text{ for all } x \in \Omega\}$$

is the dual cone of Ω. When T is single valued, the problem MCP(T, Ω) is usually called the nonlinear complementarity problem NCP(T, Ω).

In last decade, the multivalued complementarity problem was studied by several authors via the notion of exceptional families of elements (EFE for short); see [1–6] and references there in. For a survey along this direction, see [7,8].

For given a multivalued mapping T from a closed convex cone $\Omega \subset X$ into X^*, a family of elements $\{x_r : r > 0\}$ in Ω is called an EFE for (T, Ω) if the following conditions are satisfied :

(1) $\lim_{r\to\infty} \|x_r\| = \infty$.

(2) For every $r > 0$, there exist a $y_r \in T(x_r)$, a $z_r \in J(x_r)$ and a real number $t_r > 0$ such that $t_r z_r + y_r \in \Omega^*$ and $\langle t_r z_r + y_r, x_r \rangle = 0$, where J denotes the normalized duality mapping on X defined by

$$J(x) = \{y \in X^* : \|y\| = \|x\| \text{ and } \langle y, x \rangle = \|x\|^2\} \quad \text{for} \quad x \in X.$$

For every $r > 0$, let $\Omega_r = \{x \in \Omega : \|x\| \leq r\}$. It follows from [2, Theorem 3.1] that, in a Banach space X, if the problem MCP(T, Ω) has no solutions, and if for every $r > 0$ the problem GVI(T, Ω_r) has a solution (x_r, y_r), then $\{x_r : r > 0\}$ is an EFE for (T, Ω). When X is reflexive, Ω_r are weakly compact. This motivates the consideration of this work.

Most existence results in articles for generalized variational inequalities associated with mappings defined on weakly compact and convex sets are established under some generalized monotonicity assumption. It was claimed in [2] that, without monotonicity assumption, the regular complete continuity is the weakest continuity assumption used so far in the literature for dealing with such problems, cf. [2, Lemma 4.1]. See Sect. 2 for the definition of regular complete continuity. The purpose of this paper is to study the solvability of the problem GVI(T, Ω) for any given multivalued mapping T from a convex and weakly compact subset Ω of X into X^* with some continuity assumption that is weaker than the regular complete continuity.

To weaken the regular complete continuity, in Sect. 2, we first introduce a continuity assumption which is slightly weaker than the regular complete continuity, called the (HD) condition. Some equivalent statements of the (HD) condition are given in Theorem 2.2. By relaxing the (HD) condition further, we obtain a much weaker continuity assumption, called the (WSC) condition.

In Sect. 3, we derive an existence result for generalized variational inequalities associated with mappings satisfying the (WSC) condition on weakly compact subsets of X. With this result, we prove that if a mapping on a closed convex cone in a reflexive Banach space satisfies the (WSC) condition, then either the corresponding complementarity problem has a solution or there is an EFE for the mapping. By modifying the (WSC) condition slightly, some more existence results are derived.

2 The (WSC) condition

This section is used to introduce some conditions that are weaker than the regular complete continuity. To proceed, we need some notations. We use X_b^* for the space X^* equipped with the norm topology, and X_s^* for the space X^* equipped with the weak-star topology.

For given a sequence $\{x_n\}_{n=1}^{\infty}$ in X, we write $x_n \longrightarrow x \in X$ when $\{x_n\}_{n=1}^{\infty}$ converges to x in norm, and write $x_n \overset{w}{\longrightarrow} x$ when $\{x_n\}_{n=1}^{\infty}$ converges weakly to x. For given a sequence $\{y_n\}_{n=1}^{\infty}$ in X^*, we write $y_n \longrightarrow y \in X^*$ when $\{y_n\}_{n=1}^{\infty}$ is convergent to y in X_b^*, and write $y_n \overset{w^*}{\longrightarrow} y$ when $\{y_n\}_{n=1}^{\infty}$ converges to y in X_s^*.

A multivalued mapping T from a nonempty convex set $\Omega \subset X$ into X^* is regular completely upper semicontinuous [2] if the following conditions are satisfied.

(R1) T maps every bounded subset of Ω into a relatively compact subset of X_b^*.

(R2) If $\{(x_n, y_n)\}_{n=1}^{\infty}$ is a sequence in $\mathcal{G}_T$ with $x_n \overset{w}{\longrightarrow} x \in \Omega$ and $y_n \longrightarrow y \in X^*$, then $y \in T(x)$.

A single valued mapping $T : \Omega \longrightarrow X^*$ is regular completely continuous if the multivalued mapping $x \longmapsto \{T(x)\}$ is regular completely upper semicontinuous. In this case, the condition (R2) becomes :

(R2)′ For any sequence $\{x_n\}_{n=1}^{\infty}$ in Ω, if $x_n \overset{w}{\longrightarrow} x \in \Omega$, then $T(x_n) \longrightarrow T(x)$.

To weaken the regular complete continuity, we start with the following necessary condition for multivalued mappings being regular completely upper semicontinuous.

Theorem 2.1 *Let T be a multivalued mapping from a nonempty convex subset Ω of X into X^*. If T is regular completely upper semicontinuous, then T maps every weakly compact subset of Ω onto a compact subset of X_b^*. Consequently, $T(x)$ is compact in X_b^* for every $x \in \Omega$.*

Proof Let E be any weakly compact subset of Ω, and we prove that $T(E)$ is sequentially compact in X_b^*. Consider any sequence $\{y_n\}_{n=1}^{\infty}$ in $T(E)$. For every n, let $x_n \in E$ be such that $y_n \in T(x_n)$. It follows from Eberlein-Smulian Theorem [10, 2.8.6, p. 248] that $\{x_n\}_{n=1}^{\infty}$ has a subsequence weakly convergent to some point $x \in E$. Replacing $\{x_n\}_{n=1}^{\infty}$ by this subsequence, we assume that $x_n \overset{w}{\longrightarrow} x$. Since $T(E)$ is relatively compact in X_b^*, there is a subsequence $\{y_{p(n)}\}_{n=1}^{\infty}$ of $\{y_n\}_{n=1}^{\infty}$ such that $y_{p(n)} \longrightarrow y \in X^*$. Now, the condition (R2) implies $y \in T(x) \subset T(E)$. The proof is complete. □

A multivalued mapping T from a nonempty set $\Omega \subset X$ into X^* is said to have compact values in X_b^* (resp. in X_s^*) if $T(x)$ is compact in X_b^* (resp. in X_s^*) for every $x \in \Omega$.

In view of Theorem 2.1, we first slightly weaken the regular complete continuity by considering multivalued mappings which map weakly compact sets onto compact sets of X_b^*. Such multivalued mappings satisfying the condition (R2) can be characterized by the distances from points of $T(x_n)$ to that of $T(x)$. For describing the characterization, we write

$$\delta(A, B) = \sup\{\text{dist}(a, B) : a \in A\}$$

for any nonempty subsets A and B of X, where

$$\text{dist}(a, B) = \inf\{\|a - b\| : b \in B\}.$$

If A is a compact subset of X, then $\delta(A, B) = \text{dist}(a, B)$ for some $a \in A$. Note that, when A and B are compact subsets of X, the number $\max\{\delta(A, B), \delta(B, A)\}$ is the Hausdorff distance between A and B.

Theorem 2.2 *Let T be a multivalued mapping from a nonempty convex subset Ω of X into X^*. If T has compact values in X_b^*, then the following statements are equivalent.*

(i) *T satisfies the condition* (R2) *and maps every weakly compact subset of Ω onto a compact subset of X_b^*.*

(ii) *T satisfies the* (HD) *condition : If $\{x_n\}_{n=1}^{\infty}$ is a sequence in Ω with $x_n \xrightarrow{w} x \in \Omega$, then $\liminf_{n \to \infty} \delta(T(x_n), T(x)) = 0$.*

(iii) *If $\{x_n\}_{n=1}^{\infty}$ is a sequence in Ω with $x_n \xrightarrow{w} x \in \Omega$, and if $y_n \in T(x_n)$, then $\{y_n\}_{n=1}^{\infty}$ has a subsequence $\{y_{p(n)}\}_{n=1}^{\infty}$ such that $y_{p(n)} \longrightarrow y$ for some $y \in T(x)$.*

Proof First, we prove that T satisfies the (HD) condition if and only if it satisfies the condition (iii). Let $\{x_n\}_{n=1}^{\infty}$ be an arbitrary sequence in Ω with $x_n \xrightarrow{w} x \in \Omega$. For every integer $n > 0$, let $w_n \in T(x_n)$ be such that

$$\text{dist}(w_n, T(x)) = \delta(T(x_n), T(x)).$$

If T satisfies the condition (iii), then $\{w_n\}_{n=1}^{\infty}$ has a subsequence $\{w_{p(n)}\}_{n=1}^{\infty}$ convergent in X_b^* to some point of $T(x)$. This implies that

$$\lim_{n \to \infty} \text{dist}(w_{p(n)}, T(x)) = 0 \quad \text{and} \quad \liminf_{n \to \infty} \delta(T(x_n), T(x)) = 0.$$

Conversely, assume that T satisfies the (HD) condition, and consider any $y_n \in T(x_n)$ for every n. Since $\text{dist}(y_n, T(x)) \leq \text{dist}(w_n, T(x))$, there is a subsequence $\{y_{p(n)}\}_{n=1}^{\infty}$ of $\{y_n\}_{n=1}^{\infty}$ such that

$$\lim_{n \to \infty} \text{dist}(y_{p(n)}, T(x)) = 0.$$

For every n, let $z_n \in T(x)$ be such that

$$\text{dist}(y_n, T(x)) = \|y_n - z_n\|.$$

By the compactness of $T(x)$ in X_b^*, $\{z_{p(n)}\}_{n=1}^{\infty}$ has a subsequence $\{z_{q(n)}\}_{n=1}^{\infty}$ such that $z_{q(n)} \longrightarrow y \in T(x)$. This implies that $\|y_{q(n)} - y\| \longrightarrow 0$.

Next, we prove that T satisfies the (HD) condition if the statement (i) holds. For any given weakly convergent sequence $\{x_n\}_{n=1}^{\infty}$ in Ω with the limit $x \in \Omega$, the set E consisting of all x_n together with x is clearly a weakly compact subset of Ω. For every integer $n > 0$, let $w_n \in T(x_n)$ be given as above. Since $T(E)$ is a compact subset of X_b^*, $\{w_n\}_{n=1}^{\infty}$ has a subsequence $\{w_{p(n)}\}_{n=1}^{\infty}$ such that $w_{p(n)} \longrightarrow y \in X^*$. Now, the condition (R2) yields $y \in T(x)$ and $\text{dist}(w_{p(n)}, T(x)) \longrightarrow 0$. Therefore,

$$\liminf_{n \to \infty} \delta(T(x_n), T(x)) = 0.$$

Finally, we prove that the statement (i) holds if T satisfies the (HD) condition. Let E be a nonempty weakly compact subset of Ω. Let $\{y_n\}_{n=1}^{\infty}$ be any sequence in $T(E)$, and let $x_n \in E$ be such that $y_n \in T(x_n)$ for every integer $n > 0$. By Eberlein-Smulian Theorem, $\{x_n\}_{n=1}^{\infty}$ has a subsequence $\{x_{p(n)}\}_{n=1}^{\infty}$ such that $x_{p(n)} \xrightarrow{w} x \in E$. It now follows from (iii) that $\{y_n\}_{n=1}^{\infty}$ has a subsequence converging in X_b^* to some point of $T(x) \subset T(E)$. This proves that $T(E)$ is compact in X_b^*.

To prove that T satisfies the (R2) condition, let $\{x_n\}_{n=1}^{\infty}$ be any weakly convergent sequence in Ω with the limit $x \in \Omega$, and let $y_n \in T(x_n)$ be such that $y_n \longrightarrow y \in X^*$. Since

$$\text{dist}(y_n, T(x)) \leq \delta(T(x_n), T(x)) \quad \text{for every } n,$$

there is a subsequence $\{y_{p(n)}\}_{n=1}^{\infty}$ of $\{y_n\}_{n=1}^{\infty}$ such that

$$\lim_{n \to \infty} \text{dist}(y_{p(n)}, T(x)) = 0.$$

For every $n > 0$, let $z_n \in T(x)$ be such that

$$\text{dist}(y_n, T(x)) = \|y_n - z_n\|.$$

The compactness of $T(x)$ in X_b^* implies that $\{z_{p(n)}\}_{n=1}^{\infty}$ has a subsequence $\{z_{q(n)}\}_{n=1}^{\infty}$ such that $z_{q(n)} \longrightarrow z \in T(x)$. Thus, $\|y_{q(n)} - z\| \longrightarrow 0$ and $y = z \in T(x)$. □

Remark 2.1 A multivalued mapping may not have compact values in X_b^* when it satisfies the (HD) condition. For an example, consider any constant multivalued mapping T from Ω into X^* with its value a non-compact subset of X_b^*.

As before, a single valued mapping $T : \Omega \longrightarrow X^*$ is said to satisfy the (HD) condition if the multivalued mapping $x \longmapsto \{T(x)\}$ does, i.e., for any sequence $\{x_n\}_{n=1}^{\infty}$ in Ω,

$$x_n \xrightarrow{w} x \in \Omega \Longrightarrow \liminf_{n \to \infty} \|T(x_n) - T(x)\| = 0.$$

As an immediate consequence of Theorem 2.2, we conclude that a single valued mapping $T : \Omega \longrightarrow X^*$ satisfies the (HD) condition if and only if T maps every weakly compact subset of Ω onto a compact subset of X_b^* and satisfies the condition (R2)$'$.

Remark 2.2 A single valued mapping $T : \Omega \longrightarrow X^*$ may not satisfy the (HD) condition when it maps every weakly compact subset of Ω onto a compact subset of X_b^*. For an example, choose any fixed $y \in X^* \setminus \{0\}$, and consider the mapping $T : X \longrightarrow X^*$ defined by $T(0) = 0$ and $T(x) = y$ whenever $x \neq 0$. Clearly, T maps every subset of X onto a compact subset of X_b^*. Now, choose any sequence $\{x_n\}_{n=1}^{\infty}$ in $X \setminus \{0\}$ such that $x_n \longrightarrow 0$. Then $\liminf_{n \to \infty} \|T(x_n) - T(0)\| = \|y\| > 0$.

Theorem 2.2 says that a multivalued mapping satisfying the (HD) condition is almost regular completely upper semicontinuous. To get a weaker condition, we consider any mapping $T : \Omega \longrightarrow 2^{X^*}$ satisfying the (HD) condition, and assume that T has compact values in X_b^*. From Theorem 2.2 we conclude that if $\{(x_n, y_n)\}_{n=1}^{\infty}$ is a sequence in $\mathcal{G}_T$ with $x_n \xrightarrow{w} x \in \Omega$, then there is a subsequence $\{y_{p(n)}\}_{n=1}^{\infty}$ of $\{y_n\}_{n=1}^{\infty}$ such that $y_{p(n)} \longrightarrow y \in T(x)$. Consequently,

$$\lim_{n \to \infty} \langle y_{p(n)}, u - x_{p(n)} \rangle = \langle y, u - x \rangle \quad \text{for all} \quad u \in \Omega.$$

This motivates the following consideration.

2.1 The (WSC) condition

A multivalued mapping T from a nonempty convex subset Ω of X into X^* will be said to satisfy the (WSC) condition if for any sequence $\{(x_n, y_n)\}_{n=1}^{\infty}$ in $\mathcal{G}_T$ with $x_n \xrightarrow{w} x \in \Omega$, there exists $y \in T(x)$ such that

$$\liminf_{n \to \infty} \langle y_n, u - x_n \rangle \leq \langle y, u - x \rangle \quad \text{for all} \quad u \in \Omega.$$

We have proved that if T has compact values in X_b^* and satisfies the (HD) condition then T satisfies the (WSC) condition.

A single valued mapping $T : \Omega \longrightarrow X^*$ satisfies the (WSC) condition if for any weakly convergent sequence $\{x_n\}_{n=1}^{\infty}$ in Ω with the weak limit $x \in \Omega$,

$$\liminf_{n \to \infty} \langle T(x_n),\, u - x_n \rangle \leq \langle T(x),\, u - x \rangle \quad \text{for all} \quad u \in \Omega.$$

Clearly, a single valued mapping satisfies the (WSC) condition if it satisfies the (HD) condition.

The following example shows that the (WSC) condition is realy weaker than the (HD) condition.

Example 2.1 Let X denote an infinite dimensional Hilbert space equipped with the inner product $\langle \cdot,\, \cdot \rangle$, and let $e \in X$ be an arbitrary unit vector. Let $T : X \longrightarrow X$ be the continuous linear mapping defined by $T(x) = x' - \lambda e$ whenever $x = x' + \lambda e \in X$ with $\lambda = \langle x,\, e \rangle$. We shall prove that the restriction of T to any nonempty convex set $\Omega \subset X$ satisfies the (WSC) condition, but does not satisfy the (HD) condition whenever Ω has a nonempty interior.

Let $\{x_n\}_{n=1}^{\infty}$ be any weakly convergent sequence in X with the weak limit x, and write $x_n = x'_n + \lambda_n e$ and $x = x' + \lambda e$, where $\lambda_n = \langle x_n,\, e \rangle$ and $\lambda = \langle x,\, e \rangle$. Note that $\lim_{n \to \infty} \lambda_n = \lambda$, $x'_n \stackrel{w}{\longrightarrow} x'$ and $T(x_n) \stackrel{w}{\longrightarrow} T(x)$. For every n, we have

$$\langle T(x_n),\, u - x_n \rangle \;=\; \langle T(x_n),\, u \rangle \;+\; \lambda_n^2 - \|x'_n\|^2.$$

Let $\{x_{p(n)}\}_{n=1}^{\infty}$ be any subsequence of $\{x_n\}_{n=1}^{\infty}$ with $\lim_{n \to \infty} \|x'_{p(n)}\| = s$. Then

$$s \geq \liminf_{n \to \infty} \|x'_n\| \geq \|x'\|$$

and

$$\lim_{n \to \infty} \langle T(x_{p(n)}), u - x_{p(n)} \rangle \;=\; \langle T(x),\, u \rangle \;+\; \lambda^2 - s^2 \leq \langle T(x),\, u - x \rangle.$$

This implies that

$$\liminf_{n \to \infty} \langle T(x_n), u - x_n \rangle \leq \langle T(x),\, u - x \rangle.$$

Therefore, the restriction of T to any convex set $\Omega \subset X$ satisfies the (WSC) condition.

Now, we assume that Ω has a nonempty interior. There exist $r > 0$ and $x_0 \in \Omega$ such that

$$B \;=\; \{x \in X : \|x - x_0\| \leq r\} \subset \Omega.$$

Note that $T(B)$ is a closed ball in X with nonempty interior since T is a unitary linear operator on X. Since B is weakly compact and $T(B)$ is not compact, it follows from Theorem 2.2 that T does not satisfy the (HD) condition.

The above example also shows that a mapping satisfying the (WSC) condition may not map weakly compacts set into compact sets. By giving an example, we prove below that a mapping may not satisfy the (WSC) condition when it maps every weakly compact set onto a compact set.

Example 2.2 Let X and e be given in Example 2.1, and let

$$\Omega \;=\; \{x' + \lambda e \in X : \langle x',\, e \rangle = 0\,,\, \lambda \in \mathbb{R} \text{ and } \lambda \geq \|x'\|\}.$$

Consider the mapping $T : \Omega \longrightarrow X$ defined by

$$T(0) \;=\; 0 \quad \text{and} \quad T(x' + \lambda e) \;=\; e \quad \text{whenever} \quad \lambda > 0.$$

It is clear that T maps every weakly compact subset of Ω onto a compact subset of X. For every integer $n > 0$, let $x_n = \lambda_n e$, where $\{\lambda_n\}_{n=1}^{\infty}$ is a sequence of positive numbers with $\lambda_n \longrightarrow 0$. Clearly, $x_n \longrightarrow 0$. For $u = u' + \alpha e \in \Omega \setminus \{0\}$,

$$\lim_{n \to \infty} \langle T(x_n), u - 0\rangle = \langle e, u\rangle = \alpha > 0 = \langle T(0), u - 0\rangle.$$

This proves that T does not satisfy the (WSC) condition.

3 Existence results for GVI

The main work of this section is to prove the following theorem.

Theorem 3.1 *Let T be a multivalued mapping from a nonempty weakly compact and convex set $\Omega \subset X$ into X^*. Then the problem* GVI(T, Ω) *has a solution if T has nonempty convex and compact values in X_s^*, and T satisfies the* (WSC) *condition.*

Theorem 3.1 generalizes [2, Lemma 4.1] in three folds.

(1) In Lemma 4.1 of [2], X is assumed to be a reflexive Banach space; while Theorem 3.1 holds for any normed space.
(2) The (WSC) condition is much weaker than regular complete continuity.
(3) Under the assumption of [2, Lemma 4.1], the image of the mapping in consideration must be compact in X_b^*; see Theorem 2.2. However, in Theorem 3.1, the compactness assumption on the image is not needed; see Example 2.1.

To simplify notations, we first prove Theorem 3.1 for the case where Ω is compact.

Lemma 3.1 *Let T be a multivalued mapping from a nonempty compact and convex set $\Omega \subset X$ into X^*. Then the problem* GVI(T, Ω) *has a solution if T has nonempty convex and compact values in X_s^*, and T satisfies the* (WSC) *condition.*

Proof The assertion follows from [9, Theorem 3.2] if for every $u \in \Omega$, the set

$$S_u = \{x \in \Omega : \langle y, u - x\rangle \geq 0 \text{ for some } y \in T(x)\}$$

is closed. Note that every S_u is nonempty. Let $\{x_n\}_{n=1}^{\infty}$ be any sequence in S_u such that $x_n \longrightarrow x \in \Omega$, and let $y_n \in T(x_n)$ be such that $\langle y_n, u - x_n\rangle \geq 0$. The (WSC) condition implies that there exists $y \in T(x)$ such that

$$\langle y, u - x\rangle \geq \liminf_{n \to \infty} \langle y_n, u - x_n\rangle \geq 0.$$

The proof is complete. □

Proof of Theorem 3.1. Let $\mathcal{F}$ denote the family of all nonempty finite subsets of Ω. It follows from Lemma 3.1 that for every $E \in \mathcal{F}$,

$$S_E = \{x \in \Omega : \text{there exists } y \in T(x) \text{ such that } \langle y, u - x\rangle \geq 0 \text{ for all } u \in \mathrm{co}(E)\} \neq \emptyset.$$

where co(E) denotes the convex hull of E. Let $\overline{S}_E^w$ denote the weak closure of S_E. Observe that the family $\{\overline{S}_E^w : E \in \mathcal{F}\}$ has the finite intersection property. Since Ω is weakly compact, we obtain

$$S = \bigcap_{E \in \mathcal{F}} \overline{S}_E^w \neq \emptyset.$$

Choose any fixed $\widehat{x} \in S$. For every $E \in \mathcal{F}$, we write $\widehat{E} = E \cup \{\widehat{x}\}$ and

$$D_E = \{y \in T(\widehat{x}) : \langle y, u - \widehat{x} \rangle \geq 0 \text{ for all } u \in \text{co}(\widehat{E})\}.$$

We claim that

$$D = \bigcap_{E \in \mathcal{F}} D_E \neq \emptyset.$$

This claim will complete the proof. Indeed, if $\widehat{y} \in D$, then

$$\langle \widehat{y}, (1-t)\widehat{x} + tu - \widehat{x} \rangle \geq 0 \quad \text{for every } u \in \Omega \text{ and for } 0 \leq t \leq 1.$$

In particular, $\langle \widehat{y}, u - \widehat{x} \rangle \geq 0$. This proves that $(\widehat{x}, \widehat{y})$ is a solution of $\text{GVI}(T, \Omega)$.

For the proof of the claim, we first prove that every D_E is closed in X_s^*. If $\{y_\alpha\}$ is a net in D_E with $y_\alpha \xrightarrow{w^*} y \in X^*$, then for $u \in \text{co}(\widehat{E})$,

$$\langle y, u - \widehat{x} \rangle = \lim_{\alpha} \langle y_\alpha, u - \widehat{x} \rangle \geq 0.$$

The compactness of $T(\widehat{x})$ implies that $y \in T(\widehat{x})$ and $y \in D_E$.

Note that $D_{E_2} \subset D_{E_1}$ whenever $E_1, E_2 \in \mathcal{F}$ with $E_1 \subset E_2$. By the compactness of $T(\widehat{x})$, the claim will follow if $D_E \neq \emptyset$ for every $E \in \mathcal{F}$. Since $\widehat{x} \in \overline{S}^w_{\widehat{E}}$, the weak compactness of $\overline{S}^w_{\widehat{E}}$ implies that there is a sequence $\{x_n\}_{n=1}^{\infty}$ in $S_{\widehat{E}}$ such that $x_n \xrightarrow{w} \widehat{x}$. For every n, let $y_n \in T(x_n)$ be such that $\langle y_n, u - x_n \rangle \geq 0$ for all $u \in \text{co}(\widehat{E})$. The (WSC) condition implies that there exists $y \in T(\widehat{x})$ such that for all $u \in \text{co}(\widehat{E})$,

$$\langle y, u - \widehat{x} \rangle \geq \liminf_{n \to \infty} \langle y_n, u - x_n \rangle \geq 0.$$

This proves that $y \in D_E$. □

Corollary 3.1 *Let Ω be a nonempty weakly compact and convex subset of X. If a mapping $T : \Omega \longrightarrow X^*$ satisfies the* (WSC) *condition, then the problem* $\text{VI}(T, \Omega)$ *has a solution.*

Corollary 3.2 *Let X be a reflexive Banach space, and let T be a multivalued mapping from a closed convex cone $\Omega \subset X$ into X^*. Assume that T has nonempty convex and compact values in X_s^*, and T satisfies the* (WSC) *condition. Then either the problem* $\text{MCP}(T, \Omega)$ *has a solution, or there is an EFE for (T, Ω).*

Corollary 3.3 *Let Ω be a closed convex cone in a reflexive Banach space X. If a mapping $T : \Omega \longrightarrow X^*$ satisfies the* (WSC) *condition, then either the problem* $\text{NCP}(T, \Omega)$ *has a solution, or there is an EFE for (T, Ω).*

By slightly modifying the (WSC) condition, we obtain another existence result as given below. For any nonempty convex set $\Omega \subset X$, let Ω_w denote the weak topology on Ω.

Theorem 3.2 *Let T be a multivalued mapping from a nonempty weakly compact and convex set $\Omega \subset X$ into X^* with nonempty values. Then the problem* $\text{GVI}(T, \Omega)$ *has a solution if the following conditions are satisfied.*

(i) *T has closed and convex values in X_s^*.*
(ii) *$T(\Omega)$ is a compact subset of X_s^*.*
(iii) *$\mathcal{G}_T$ is closed in $\Omega_w \times X_s^*$.*
(iv) *If $\{(x_n, y_n)\}_{n=1}^{\infty}$ is a sequence in $\mathcal{G}_T$ with $x_n \xrightarrow{w} x \in \Omega$, then $\liminf_{n \to \infty} \langle y_n, x - x_n \rangle \leq 0$.*

Proof The conditions (i) and (ii) imply that T has compact values in X_s^*. The assertion will follow if T satisfies the (WSC) condition. Consider any sequence $\{(x_n, y_n)\}_{n=1}^{\infty}$ in $\mathcal{G}_T$ with $x_n \xrightarrow{w} x \in \Omega$. With the condition (iv), $\{y_n\}_{n=1}^{\infty}$ has a subsequence $\{y_{p(n)}\}_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty}\langle y_{p(n)},\ x - x_{p(n)}\rangle = \liminf_{n\to\infty}\langle y_n,\ x - x_n\rangle \le 0.$$

By the condition (ii), there is a subnet $\{y_\alpha\}$ of $\{y_{p(n)}\}_{n=1}^{\infty}$ such that $y_\alpha \xrightarrow{w^*} y \in T(\Omega)$. The condition (iii) yields $y \in T(x)$. For every $u \in \Omega$,

$$\langle y_\alpha,\ u - x_\alpha\rangle - \langle y,\ u - x\rangle = \langle y_\alpha - y,\ u - x\rangle + \langle y_\alpha,\ x - x_\alpha\rangle$$

and

$$\begin{aligned}\liminf_{n\to\infty}\langle y_n,\ u - x_n\rangle &\le \liminf_{n\to\infty}\langle y_{p(n)},\ u - x_{p(n)}\rangle\\ &\le \liminf_{\alpha}\langle y_\alpha,\ u - x_\alpha\rangle\\ &= \langle y,\ u - x\rangle + \lim_{\alpha}\langle y_\alpha,\ x - x_\alpha\rangle\\ &= \langle y,\ u - x\rangle + \lim_{n\to\infty}\langle y_{p(n)},\ x - x_{p(n)}\rangle\\ &\le \langle y,\ u - x\rangle\end{aligned}$$

This proves that T satisfies the (WSC) condition. □

Corollary 3.4 *Let X be a reflexive Banach space, and let T be a multivalued mapping from a closed convex cone $\Omega \subset X$ into X^*. Assume that the following conditions are satisfied.*

(i) *T has closed and convex values in X_s^*.*
(ii) *T maps every weakly compact subset of Ω onto a compact subset of X_s^*.*
(iii) *$\mathcal{G}_T$ is closed in $\Omega_w \times X_s^*$.*
(iv) *If $\{(x_n, y_n)\}_{n=1}^{\infty}$ is a sequence in $\mathcal{G}_T$ with $x_n \xrightarrow{w} x \in \Omega$, then $\liminf_{n\to\infty}\langle y_n,\ x - x_n\rangle \le 0$.*

Then either the problem MCP$(T,\ \Omega)$ *has a solution, or there is an EFE for* $(T,\ \Omega)$.

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